

On uniquely 3-colorable plane graphs without prescribed adjacent faces ¹

Ze-peng LI

School of Electronics Engineering and Computer Science
Key Laboratory of High Confidence Software
Technologies of Ministry of Education,
Peking University, Beijing 100871, China

Naoki MATSUMOTO

Graduate School of Environment and Information
Sciences Yokohama National University,
Yokohama, Japan

En-qiang ZHU

School of Electronics Engineering and Computer Science
Key Laboratory of High Confidence Software
Technologies of Ministry of Education,
Peking University, Beijing 100871, China

Jin XU

School of Electronics Engineering and Computer Science
Key Laboratory of High Confidence Software
Technologies of Ministry of Education,
Peking University, Beijing 100871, China

Tommy JENSEN

Kyungpook National University
1370 Sankyuk-dong Buk-gu Daegu, 702701, Korea

Abstract

A graph G is *uniquely k -colorable* if the chromatic number of G is k and G has only one k -coloring up to permutation of the colors. For a plane graph G , two faces f_1 and f_2 of G are *adjacent (i, j) -faces* if $d(f_1) = i$, $d(f_2) = j$ and f_1 and f_2 have a common edge, where $d(f)$ is the degree of a face f . In this paper, we prove that every uniquely 3-colorable plane graph has adjacent $(3, k)$ -faces, where $k \leq 5$. The bound 5 for k is best possible. Furthermore, we prove that there exist a class of uniquely 3-colorable plane graphs having neither adjacent $(3, i)$ -faces nor adjacent $(3, j)$ -faces, where $i, j \in \{3, 4, 5\}$ and $i \neq j$. One of our constructions implies that

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there exist an infinite family of edge-critical uniquely 3-colorable plane graphs with n vertices and $\frac{7}{3}n - \frac{14}{3}$ edges, where $n(\geq 11)$ is odd and $n \equiv 2 \pmod{3}$.

Keywords plane graph; unique coloring; uniquely 3-colorable plane graph; construction; adjacent (i, j) -faces

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1 Introduction

For a plane graph G , $V(G)$, $E(G)$ and $F(G)$ are the sets of vertices, edges and faces of G , respectively. The degree of a vertex $v \in V(G)$, denoted by $d_G(v)$, is the number of neighbors of v in G . The degree of a face $f \in F(G)$, denoted by $d_G(f)$, is the number of edges in its boundary, cut edges being counted twice. When no confusion can arise, $d_G(v)$ and $d_G(f)$ are simplified by $d(v)$ and $d(f)$, respectively. A face f is a k -face if $d(f) = k$ and a k^+ -face if $d(f) \geq k$. Two faces f_1 and f_2 of G are adjacent (i, j) -faces if $d(f_1) = i$, $d(f_2) = j$ and f_1 and f_2 have at least one common edge. Two distinct paths of G are *internally disjoint* if they have no internal vertices in common.

A graph G is *uniquely k -colorable* if $\chi(G) = k$ and G has only one k -coloring up to permutation of the colors, where the coloring is called a *unique k -coloring* of G . In other words, all k -colorings of G induce the same partition of $V(G)$ into k independent sets, in which an independent set is called a *color class* of G . In addition, uniquely colorable graphs may be defined in terms of their chromatic polynomials, which initiated by Birkhoff [2] for planar graphs in 1912, and for general graphs by Whitney [11] in 1932. Because a graph G is uniquely k -colorable if and only if its chromatic polynomial is $k!$. For a discussion of chromatic polynomials, see Read [10].

Uniquely colorable graphs were first studied by Harary and Cartwright [6] in 1968. They proved the following theorem.

Theorem 1.1. (Harary and Cartwright [6]) *Let G be a uniquely k -colorable graph. Then for any unique k -coloring of G , the subgraph induced by the union of any two color classes is connected.*

As a corollary of Theorem 1.1, it can be seen that a uniquely k -colorable graph G has at least $(k-1)|V(G)| - \binom{k}{2}$ edges. There are many references on uniquely colorable graphs [5, 7, 3].

Chartrand and Geller [5] in 1969 started to study uniquely colorable planar graphs. They proved that uniquely 3-colorable planar graphs with at least 4 vertices contain at least two triangles, uniquely 4-colorable planar graphs are maximal planar graphs, and uniquely 5-colorable planar graphs do not exist. Aksionov [1] in 1977 improved the lower bound for the number of triangles in a uniquely 3-colorable planar graph. He proved that a uniquely 3-colorable planar graph with at least 5 vertices contains at least 3 triangles and gave a complete description of uniquely 3-colorable planar graphs containing exactly 3 triangles.

Let G be a uniquely k -colorable graph, G is *edge-critical* if $G - e$ is not uniquely k -colorable for any edge $e \in E(G)$. Obviously, if a uniquely

k -colorable graph G has exactly $(k-1)|V(G)| - \binom{k}{2}$ edges, then G is edge-critical. Mel'nikov and Steinberg [9] in 1977 asked to find an exact upper bound for the number of edges in an edge-critical uniquely 3-colorable planar graph with n vertices. Recently, Matsumoto [8] proved that an edge-critical uniquely 3-colorable planar graph has at most $\frac{8}{3}n - \frac{17}{3}$ edges and constructed an infinite family of edge-critical uniquely 3-colorable planar graphs with n vertices and $\frac{9}{4}n - 6$ edges, where $n \equiv 0 \pmod{4}$.

In this paper, we mainly prove Theorem 1.2.

Theorem 1.2. *If G is a uniquely 3-colorable plane graph, then G has adjacent $(3, k)$ -faces, where $k \leq 5$. The bound 5 for k is best possible.*

Furthermore, by using constructions, we prove that there exist uniquely 3-colorable plane graphs having neither adjacent $(3, i)$ -faces nor adjacent $(3, j)$ -faces, where $i, j \in \{3, 4, 5\}$ and $i \neq j$. One of our constructions implies that there exist an infinite family of edge-critical uniquely 3-colorable plane graphs with n vertices and $\frac{7}{3}n - \frac{14}{3}$ edges, where $n(\geq 11)$ is odd and $n \equiv 2 \pmod{3}$.

2 Proof of Theorem 1.2

Lemma 2.1. *Let G be a plane graph with 3-faces. If G has no adjacent $(3, k)$ -faces, where $k \leq 5$, then $|E(G)| \geq 2|F(G)|$.*

Proof. We prove this by using a simple charging scheme. Since G has no adjacent $(3, k)$ -faces when $k \leq 5$, for any edge e incident to a 3-face f , e is incident to a face of degree at least 6. Let $ch(f) = d(f)$ for any face $f \in F(G)$ and we call $ch(f)$ the initial charge of the face f . Let initial charges in G be redistributed according to the following rule.

Rule: For each 3-face f of G and each edge e incident with f , the 6^+ -face incident with e sends $\frac{1}{3}$ charge to f through e .

Denote by $ch'(f)$ the charge of a face $f \in F(G)$ after applying redistributed Rule. Then

$$\sum_{f \in F(G)} ch'(f) = \sum_{f \in F(G)} ch(f) = \sum_{f \in F(G)} d(f) = 2|E(G)| \quad (1)$$

On the other hand, for any 3-face f of G , since the degree of each face adjacent to f is at least 6, then by the redistributed Rule, $ch'(f) = ch(f) + 3 \cdot \frac{1}{3} = d(f) + 1 = 4$. For any 4-face or 5-face f of G , $ch'(f) = ch(f) = d(f) \geq 4$. For any 6^+ -face f of G , since f is incident to at most $d(f)$ edges each of which is incident to a 3-face,

then $ch'(f) \geq ch(f) - \frac{1}{3}d(f) = \frac{2}{3}d(f) \geq 4$. Therefore, we have

$$\sum_{f \in F(G)} ch'(f) \geq \sum_{f \in F(G)} 4 = 4|F(G)| \quad (2)$$

By the formulae (1) and (2), we have $|E(G)| \geq 2|F(G)|$. \square

Proof of Theorem 1.2 Suppose that the theorem is not true and let G be a counterexample to the theorem. Then G has at least one 3-face and no adjacent $(3, k)$ -faces, where $k \leq 5$. By Lemma 2.1, $|E(G)| \geq 2|F(G)|$. Using Euler's Formula $|V(G)| - |E(G)| + |F(G)| = 2$, we can obtain

$$|E(G)| \leq 2|V(G)| - 4.$$

Since G is uniquely 3-colorable, then by Theorem 1.1, we have $|E(G)| \geq 2|V(G)| - 3$. This is a contradiction.

Note that the graph shown in Fig. 1 is a uniquely 3-colorable plane graph having neither adjacent $(3, 3)$ -faces nor adjacent $(3, 4)$ -faces. Therefore, the bound 5 for k is best possible. \square

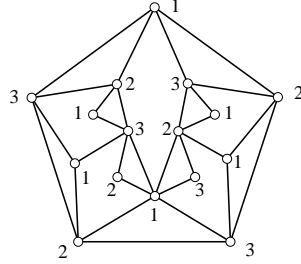


Figure 1: A uniquely 3-colorable plane graph having neither adjacent $(3, 3)$ -faces nor adjacent $(3, 4)$ -faces

Remark. By piecing together more copies of the plane graph in Fig. 1, one can construct an infinite class of uniquely 3-colorable plane graphs having neither adjacent $(3, 3)$ -faces nor adjacent $(3, 4)$ -faces.

3 Construction of uniquely 3-colorable plane graphs without adjacent $(3, 3)$ -faces or adjacent $(3, 5)$ -faces

There are many classes of uniquely 3-colorable plane graphs having neither adjacent $(3, 4)$ -faces nor adjacent $(3, 5)$ -faces, such as even

maximal plane graphs (maximal plane graphs in which each vertex has even degree) and maximal outerplanar graphs with at least 6 vertices.

In this section, we construct a class of uniquely 3-colorable plane graphs having neither adjacent $(3, 3)$ -faces nor adjacent $(3, 5)$ -faces and prove that these graphs are edge-critical.

We construct a graph G_k as follows:

- (1) $V(G_k) = \{u, w, v_0, v_1, \dots, v_{3k-1}\}$;
- (2) $E(G_k) = \{v_0v_1, v_1v_2, \dots, v_{3k-2}v_{3k-1}, v_{3k-1}v_0\} \cup \{uv_i : i \equiv 1 \text{ or } 2 \pmod{3}\} \cup \{wv_i : i \equiv 0 \text{ or } 1 \pmod{3}\}$, where k is odd and $k \geq 3$. (See an example G_3 shown in Fig. 2.)

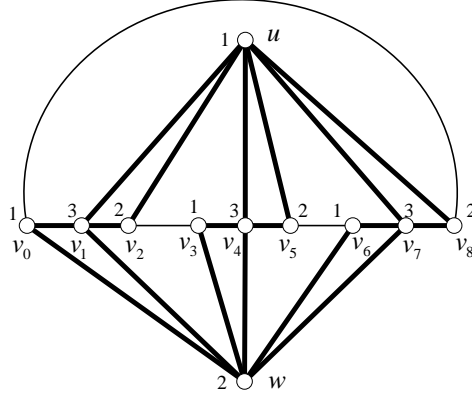


Figure 2: An example G_3

Theorem 3.1. *For any odd k with $k \geq 3$, G_k is uniquely 3-colorable.*

Proof. Let f be any 3-coloring of G_k . Since $v_0v_1 \dots v_{3k-1}v_0$ is a cycle of odd length and each v_i is adjacent to u or w , we have $f(u) \neq f(w)$. Without loss of generality, let $f(u) = 1$ and $f(w) = 2$. By the construction of G_k , we know that v_{3j+1} is adjacent to both u and w , where $j = 0, 1, \dots, k-1$. So v_{3j+1} can only receive the color 3, namely $f(v_{3j+1}) = 3$, $j = 0, 1, \dots, k-1$. Since v_{3j} is adjacent to both w and v_{3j+1} in G_k , we have $f(v_{3j}) = 1$, $j = 0, 1, \dots, k-1$. Similarly, we can obtain $f(v_{3j+2}) = 2$, $j = 0, 1, \dots, k-1$. Therefore, the 3-coloring f is uniquely decided as shown in Fig. 2 and then G_k is uniquely 3-colorable. \square

Theorem 3.2. *For any odd k with $k \geq 3$, G_k is edge-critical.*

Proof. To complete the proof it suffices to show that $G_k - e$ is not uniquely 3-colorable for any edge $e \in E(G_k)$. Let f be a uniquely

3-coloring of G_k shown in Fig. 2. Denote by E_{ij} the set of edges in G_k whose ends colored by i and j , respectively, where $1 \leq i < j \leq 3$. Namely

$$E_{ij} = \{xy : xy \in E(G_k), f(x) = i, f(y) = j\}, 1 \leq i < j \leq 3.$$

Observation 1. Both the subgraphs $G_k[E_{13}]$ and $G_k[E_{23}]$ of G_k induced by E_{13} and E_{23} are trees.

Observation 2. The subgraph $G_k[E_{12}]$ of G_k induced by E_{12} consists of k internally disjoint paths $uv_{3i-1}v_{3i}w$, where $i = 1, 2, \dots, k$.

If $e \in E_{13} \cup E_{23}$, then $G_k - e$ is not uniquely 3-colorable by Observation 1. Suppose that $e \in E_{12}$. By Observation 2, there exists a number $t \in \{1, 2, \dots, k\}$ such that $e \in \{uv_{3t-1}, v_{3t-1}v_{3t}, v_{3t}w\}$. Moreover, $G_k - e$ contains at least one vertex of degree 2. By repeatedly deleting vertices of degree 2 in $G_k - e$ we can obtain a subgraph $G_k - \{v_{3t-1}, v_{3t}\}$ of G_k . Now we prove that $G_k - \{v_{3t-1}, v_{3t}\}$ is not uniquely 3-colorable.

It can be seen that the restriction f_0 of f to the vertices of $G_k - \{v_{3t-1}, v_{3t}\}$ is a 3-coloring of $G_k - \{v_{3t-1}, v_{3t}\}$. On the other hand, $G_k - \{v_{3t-1}, v_{3t}, u, w\}$ is a path, denoted by P . Let $f'(u) = f'(w) = 1$ and alternately color the vertices of P by the other two colors. We can obtain a 3-coloring f' of $G_k - \{v_{3t-1}, v_{3t}\}$ which is distinct from f_0 . Since each 3-coloring of $G_k - \{v_{3t-1}, v_{3t}\}$ can be extended to a 3-coloring of $G_k - e$, we know that $G_k - e$ is not uniquely 3-colorable when $e \in E_{12}$.

Since $E(G_k) = E_{12} \cup E_{13} \cup E_{23}$, $G_k - e$ is not uniquely 3-colorable for any edge $e \in E(G_k)$. \square

Note that G_k has $3k+2$ vertices and $7k$ edges by the construction. From Theorem 3.2 we can obtain the following result.

Corollary 3.1. *There exist an infinite family of edge-critical uniquely 3-colorable plane graphs with n vertices and $\frac{7}{3}n - \frac{14}{3}$ edges, where $n(\geq 11)$ is odd and $n \equiv 2 \pmod{3}$.*

Denote by $size(n)$ the upper bound of the number of edges of edge-critical uniquely 3-colorable planar graphs with n vertices. Then by Corollary 3.1 and the result due to Matsumoto [8], we can obtain the following result.

Corollary 3.2. *For any odd integer n such that $n \equiv 2 \pmod{3}$ and $n \geq 11$, we have $\frac{7}{3}n - \frac{14}{3} \leq size(n) \leq \frac{8}{3}n - \frac{17}{3}$.*

4 Concluding remarks

In this paper we obtained a structural property of uniquely 3-colorable plane graphs. We proved that every uniquely 3-colorable

plane graph has adjacent $(3, k)$ -faces, where $k \leq 5$, and the bound 5 for k is best possible. Fig. 1 shows a uniquely 3-colorable plane graph having neither adjacent $(3, 3)$ -faces nor adjacent $(3, 4)$ -faces. But this plane graph is 2-connected. This prompts us to propose the following conjecture.

Conjecture 4.1. *Let G be a 3-connected uniquely 3-colorable plane graph. Then G has adjacent $(3, k)$ -faces, where $k \leq 4$.*

It can be seen that the uniquely 3-colorable plane graph G_k constructed in Section 3 is 3-connected. So if Conjecture 4.1 is true, then the bound 4 for k is best possible.

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